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An approximate solution of the parallel whistler-mode dispersion equation in a weakly relativistic plasma

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Abstract. A new approximate solution of the parallel whistler-mode dispersion equation in a weakly relativistic plasma is obtained under the assumption that the real part of the wave refractive index, N, predicted by this solution is close to the value of N predicted by the solution of the corresponding non-relativistic dispersion equation. Both the values of N and the values of the increment of wave growth or damping are closer to those obtained by numerical solution of the weakly relativistic dispersion equation than previously suggested approximations. The accuracy of the obtained solution increases with decreasing electron density, and/or anisotropy of the electron distribution function, and/or electron energy.

1. Introduction

Attention to whistler-mode waves has been mainly stimulated by the important role which they play both in space (see e.g. Omura *et al* 1991, Sazhin *et al* 1992a) and laboratory (see e.g. Guest *et al* 1990, Garner *et al* 1990, Chen 1991, Komori *et al* 1991) plasmas. Most of the theoretical studies of these waves have been based on the assumption that the plasma through which they propagate is mainly 'cold' in the sense that non-zero electron thermal velocities do not influence wave propagation (see e.g. Kennel and Petschek 1966). This assumption has been in most cases justified only by its simplicity—in general the solution of the dispersion equation for whistler-mode waves in a hot plasma requires rather complicated numerical methods (see e.g. Tsang 1984, Temme *et al* 1992, Sazhin *et al* 1992b). At the same time attempts have been made to find approximate solutions of the whistler-mode dispersion equation in a hot anisotropic plasma which would combine the simplicity of the 'cold' plasma solution and the generality of the numerical solutions (Sazhin 1992). Some of these solutions have been compared with one another and with the results of numerical analyses of the parallel whistler-mode dispersion equation in a hot anisotropic plasma, with weakly relativistic effects taken and not taken into account (Sazhin *et al* 1992b). In this paper we develop further the analysis by Sazhin *et al* (1992b) and consider one more approximate solution of the parallel whistler-mode dispersion equation in a

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weakly relativistic plasma, which is presented in the form of a perturbation of the numerical solution of the non-relativistic whistler-mode dispersion equation. As in the paper by Sazhin *et al* (1992b) we restrict ourselves to waves propagating parallel to the magnetic field which are most important for applications and simple for theoretical analysis.

In section 2 we review the main equations used in our analysis and their approximate solutions discussed by Sazhin *et al* (1992b). In section 3 we derive the new approximate solution of the dispersion equation for parallel whistler-mode waves in a hot anisotropic plasma with weakly relativistic effects taken into account. In section 4 this new solution is compared with previously obtained analytical and numerical solutions. The main results of the paper are summarized in section 5.

2. Basic equations

The dispersion equation for parallel whistler-mode waves in a weakly relativistic anisotropic plasma can be written as (cf Sazhin and Temme 1990)

$$N^2 = 1 - \frac{2X}{r}\left[\mathcal{F}_{\frac{1}{2},2} - (\mathcal{F}_{\frac{3}{2},2} - \mathcal{F}_{\frac{1}{2},2})(A_e - 1)N^2\right] \tag{1}$$

where

$$\mathcal{F}_{q,p} \equiv \mathcal{F}_{q,p}(z,a,b) = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt-(at^2/1-\mathrm{i}t)}(1-\mathrm{i}t)^{-q}(1-\mathrm{i}bt)^{-p}\,\mathrm{d}t \tag{2}$$

(the generalized Shkarofsky function), $z = 2(1-Y)/r$; $a = N^2/r$; $r = p_{0\|}^2/(m_e^2c^2)$; $b = A_e$; $X = \Pi_0^2/\omega^2$; $Y = \Omega_0/\omega$; $A_e = p_{0\perp}^2/p_{0\|}^2$; Π_0, Ω_0 and ω are the electron plasma frequency at rest, the electron gyrofrequency at rest and the wave frequency (which is complex in general) respectively, $N \equiv ck/\omega$ is the wave refractive index, m_e is the electron mass at rest, c is the velocity of light, k is the wavenumber. When deriving (1) we assumed an electron distribution function of the form

$$f(p_\perp, p_\|) = (j!\pi^{3/2}p_{0\perp}^{2j+2}p_{0\|})^{-1}p_\perp^{2j}\exp\left(-\frac{p_\perp^2}{p_{0\perp}^2} - \frac{p_\|^2}{p_{0\|}^2}\right) \tag{3}$$

where $p_{0\perp(\|)}$ is the electron thermal momentum in the direction perpendicular (parallel) to the magnetic field, $p_\perp$ and $p_\|$ are the electron momenta in the corresponding directions, $j = 0, 1, 2, \ldots$, and we restricted our analysis to the case $j = 0$ (bi-Maxwellian plasma). The generalization to $j \neq 0$ would be straightforward (Tsai *et al* 1981, Sazhin 1989). We also restricted ourselves to considering wave frequencies well above the proton gyrofrequency, so that the protons and other heavy ions could be considered as a neutralizing background, and assumed that $p_{0\|(\perp)} \ll m_e c$ (weakly relativistic approximation).

In a non-relativistic limit, $c \to \infty$, we have

$$\mathcal{F}_{q,p} = -\frac{1}{2\sqrt{a}}Z\left(\frac{z}{2\sqrt{a}}\right) \tag{4}$$

where $Z \equiv Z(\xi)$ is the non-relativistic plasma dispersion function given by

$$Z(\xi) = \mathrm{i}\sqrt{\pi}\exp(-\xi^2) - 2\int_0^\xi \exp(-\xi^2 + t^2)\,\mathrm{d}t\,. \tag{5}$$

In view of (1) and (4) the non-relativistic parallel whistler-mode dispersion equation can be written as

$$N^2 = 1 + (A_e - 1)X + \frac{X}{N\sqrt{r}}[A_e + (1 - A_e)Y]\,Z(\xi) \tag{6}$$

where $\xi = z/2\sqrt{a}$.

Solutions to equations (1) and (6) are greatly simplified if we restrict our analysis to the case of weakly growing or weakly damped waves, i.e.

$$|\gamma| \equiv |\operatorname{Im}\omega| \ll \min(\omega_0, \Omega_0 - \omega_0) \tag{7}$$

where $\omega_0 \equiv \operatorname{Re}\omega$.

The condition (7) is not restrictive for applications of our theory. If it is violated we cannot neglect non-linear effects, as we did when deriving equations (1) and (6).

In view of (7) we can present the complex equation (1) as a system of two equations

$$N^2 = 1 - \frac{2X}{r}\left[\operatorname{Re}\mathcal{F}_{\frac{1}{2},2} - \frac{\mathrm{d}\operatorname{Re}\mathcal{F}_{\frac{3}{2},2}}{\mathrm{d}z}(A_e - 1)N^2\right] \tag{8}$$

$$\tilde{\gamma} \equiv \frac{\gamma}{\omega} = \frac{X\left[\operatorname{Im}\mathcal{F}_{\frac{1}{2},2}(1 + (A_e - 1)N^2) - (A_e - 1)N^2\operatorname{Im}\mathcal{F}_{\frac{3}{2},2}\right]}{r\left[1 + (2X/r^2)\sum_{i=0}^{3}\tilde{a}_i\operatorname{Re}\mathcal{F}_{i-\frac{1}{2},2}\right]} \tag{9}$$

where

$$\tilde{a}_0 = Y + N^2[1 + Y(A_e - 1)] + N^4(A_e - 1)$$
$$\tilde{a}_1 = (A_e - 1)N^2 r - Y - 2N^2[1 + Y(A_e - 1)] - 3N^4(A_e - 1)$$
$$\tilde{a}_2 = -(A_e - 1)N^2 r + N^2[1 + Y(A_e - 1)] + 3N^4(A_e - 1)$$
$$\tilde{a}_3 = -(A_e - 1)N^4\,.$$

To simplify the notation hereafter we assume that $\omega \equiv \operatorname{Re}\omega \equiv \omega_0$.

Equation (8) describes wave propagation, while equation (9) describes wave growth ($\tilde{\gamma} > 0$) or damping ($\tilde{\gamma} < 0$) in a weakly relativistic plasma.

In a similar way we can reduce the complex non-relativistic dispersion (6) to the following system of equations:

$$N^2 = 1 + (A_e - 1)X + \frac{X}{N\sqrt{r}}[A_e + (1 - A_e)Y]\operatorname{Re}Z(\xi_0) \tag{10}$$

$$\tilde{\gamma} \equiv \frac{\gamma}{\omega} = \frac{-\sqrt{\pi}\,[A_e + (A_e - 1)Y]\exp(-\xi_0^2)}{[2(1 - Y)/X\xi_0] + \kappa\xi_0 + \operatorname{Re}Z(\xi_0)\left[A_e + \kappa\xi_0^2\right]} \tag{11}$$

where

$$\xi_0 \equiv \mathrm{Re}\,\xi \qquad \kappa = \frac{2[A_e + Y(1 - A_e)]}{Y - 1}.$$

Equation (10) describes wave propagation, while (11) describes wave growth or damping in a hot anisotropic (but non-relativistic) plasma.

In the cold plasma limit ($|\xi_0| \to \infty$) equation (10) reduces to

$$N = N_{00} \equiv \sqrt{1 + \frac{X}{Y - 1}} \tag{12}$$

while (11) in the same limit gives $\tilde{\gamma} = 0$. This means that in a cold plasma whistler-mode waves propagate without damping or growth.

In the limit $r \to 0$, but keeping terms of the order of r, we can write an approximate solution of (8) as

$$N = N_{00}\left[1 + \frac{\beta_e Y^3}{2(Y-1)^3} + \frac{\beta_e(1 + 4A_e)Y^2}{4(Y-1)^2 N_{00}^2} - \frac{\beta_e A_e Y^2}{2(Y-1)^2}\right] \tag{13}$$

where $\beta_e = 0.5\nu r$, $\nu = \Pi_0^2/\Omega_0^2$ and N_{00} is defined by (12). In the limit $N_{00}^2 \to \infty$ this expression reduces to that which could be derived from the corresponding non-relativistic dispersion equation (10).

The non-relativistic expression for $\tilde{\gamma}$ can be considerably simplified if we neglect the effects of non-zero electron temperature on wave propagation; it is then reduced to

$$\tilde{\gamma} = \frac{\sqrt{\pi}\,\xi_{00}(Y-1)\,[A_e - (A_e - 1)Y]\exp(-\xi_{00}^2)}{[2(1-Y)^2/X] + Y} \tag{14}$$

where

$$\xi_{00} = \frac{1 - Y}{N_{00}\sqrt{r}}.$$

3. An approximate solution

The solution (13) has an obvious advantage when compared with the cold plasma solution N_{00} (see (12)) as it takes into account the thermal and relativistic corrections to N_{00}. However, (13) is valid only when these corrections are small, which is not true when the wave frequency approaches the electron gyrofrequency (condition (7) should remain valid in any case). In the latter case we can look for the solution of (8) not in terms of the perturbation with respect to N_{00}, but with respect to the solution of the non-relativistic dispersion equation (10).

When $a \gg 1$ we can write (Sazhin and Temme 1990)

$$\mathrm{Re}\,\mathcal{F}_{q,p} = \frac{1}{2\sqrt{a}}\left[-\mathrm{Re}\,Z - \frac{\xi}{2\sqrt{a}}\left(\mathrm{Re}\,Z + \xi\,\mathrm{Re}\,Z'\right) + \frac{1 - q - pb}{2\sqrt{a}}\mathrm{Re}\,Z'\right]. \tag{15}$$

In view of (15), (8) can be simplified to

$$N^2 = 1 + X(A_e - 1) + \frac{X}{r\sqrt{a}}\left[\operatorname{Re} Z + \frac{\xi}{2\sqrt{a}}\left(\operatorname{Re} Z + \xi \operatorname{Re} Z'\right)\right.$$
$$\left. - \frac{1-4A_e}{4\sqrt{a}}\operatorname{Re} Z' + \frac{(A_e - 1)N^2\xi \operatorname{Re} Z}{\sqrt{a}}\right]. \tag{16}$$

We look for the solution of (16) in the form

$$N = N_{rs} \equiv N_n + \Delta N \tag{17}$$

where N_n is the solution of (10), $|\Delta N| \ll N_n$ and subscript $_{rs}$ indicates that we are looking for a simplified relativistic solution of (8).

Having substituted (17) into (16) and neglecting second-order terms with respect to ΔN we obtain

$$\Delta N = \frac{\nu Y^2 \sqrt{r}\left[-1 + (\xi^2 - 1 + 2A_e)\operatorname{Re} Z'\right]}{4N_n^3\sqrt{r} - 2\nu Y^2[A_e + Y(1-A_e)]\left[2\xi + (2\xi^2 - 1)\operatorname{Re} Z\right]} \tag{18}$$

where

$$\xi = \xi(N_n).$$

In the limiting case $|\xi| \gg 1$ expression (17) reduces to (13). In order to prove this we need to keep at least three terms in the expansion

$$\operatorname{Re} Z = -\frac{1}{\xi} - \frac{1}{2\xi^3} - \frac{3}{4\xi^5} - \cdots.$$

Expression (17) with ΔN defined by (18) gives us the required approximate solution of (8). Subtituting this solution into (9) we obtain the corresponding approximate solution of (9). Note that we cannot use expansion (15) for the analysis of (9) as this would lead to a result which would not reduce to the non-relativistic expression (11) in the case $c \to \infty$. In order to get the correct expression for γ we would require more terms in expansion (15) with respect to $a^{-1/2}$ (the denominator in (9) contains terms proportional to N^4).

4. Numerical results

In this section we compare numerical values of N and $\bar{\gamma}$ obtained from the approximations discussed in sections 2 and 3 for some values of the parameters similar to those considered by Sazhin *et al* (1992b). This comparison will complement that given by Sazhin *et al* (1992b) where only the approximations given in section 2 were considered. Our main attention will be concentrated on the approximation N_{rs}. In order to simplify further discussion we introduce the parameters N_r, N_n, N_{00}, N_{ra}, N_{rs}, $\bar{\gamma}_r$, $\bar{\gamma}_n$, $\bar{\gamma}_{ns}$ and $\bar{\gamma}_{rs}$, the meanings of which are specified in table 1 (cf the

Table 1.

Parameter	Meaning	Reference
N_r	Weakly relativistic refractive index	Equation (8)
N_n	Non-relativistic refractive index	Equation (10)
N_{00}	Refractive index in a cold plasma	Equation (12)
N_{ra}	Approximate expression for the weakly relativistic refractive index (perturbation of N_{00})	Equation (13)
N_{rs}	Approximate expression for the weakly relativistic refractive index (perturbation of N_n)	Equation (17)
$\tilde{\gamma}_r$	Weakly relativistic value of $\tilde{\gamma}$	Equation (9)
$\tilde{\gamma}_n$	Non-relativistic value of $\tilde{\gamma}$	Equation (11)
$\tilde{\gamma}_{ns}$	Simplified non-relativistic value of $\tilde{\gamma}$	Equation (14)
$\tilde{\gamma}_{rs}$	Simplified relativistic value of $\tilde{\gamma}$ (Equation (9) with $N = N_{rs}$)	Equations (9) and (17)

corresponding table presented in Sazhin *et al* 1992b). Curves will be shown only for those values of parameters for which $|\tilde{\gamma}| < 0.2$.

First we consider a relatively hot ($r = 0.01$) and dense ($\nu = 5$) plasma with $A_e = 2$. Plots of N versus $\omega/\Omega_0 (\equiv Y^{-1})$ for different approximations are shown in figure 1. As can be seen from this figure, the values of N_{rs} are closest to those of N_r at $0.5 \lesssim \omega/\Omega_0 \lesssim 0.65$. This can be seen even more clearly from figure 2 where we present the curves

$$\tilde{N}_{rs} = (N_{rs} - N_r)/N_r \tag{19}$$

$$\tilde{N}_n = (N_n - N_r)/N_r \tag{20}$$

$$\tilde{N}_{ra} = (N_{ra} - N_r)/N_r \tag{21}$$

versus ω/Ω_0 for the same values of parameters as in figure 1. As follows from figures 1 and 2, in the whole interval $0.4 \lesssim \omega/\Omega_0 \lesssim 0.69$ N_{rs} differs from N_r by not more than 1% which is sufficient for most observations.

Plots of $\tilde{\gamma}$ versus ω/Ω_0 for the same values of parameters and the same approximations as in figures 1 and 2 are shown in figure 3. As follows from figure 3, $\tilde{\gamma}_{rs}$ is the best approximation for $\tilde{\gamma}_r$ in the whole frequency range under consideration. In fact a visible deviation between the curves $\tilde{\gamma}_{rs}$ versus ω/Ω_0 and $\tilde{\gamma}_r$ versus ω/Ω_0 appears only at frequencies close to those at which $\tilde{\gamma}_{rs}$ and $\tilde{\gamma}_r$ are maximal. Both curves $\tilde{\gamma}_{rs}$ versus ω/Ω_0 and $\tilde{\gamma}_r$ versus ω/Ω_0 predict the same values of frequencies at which the waves are marginally stable.

Curves similar to those shown in figures 1 to 3 were also obtained for the following combinations of parameters:

$r = 0.01$	$\nu = 1$	$A_e = 2$
$r = 0.01$	$\nu = 0.5$	$A_e = 2$
$r = 0.004$	$\nu = 1$	$A_e = 2$
$r = 0.004$	$\nu = 1$	$A_e = 1$
$r = 0.004$	$\nu = 1$	$A_e = 3$

which are similar to those considered by Sazhin *et al* (1992b). In all these 5 cases the curves N_{rs} almost coincided with the curves N_r and the curves $\tilde{\gamma}_{rs}$ almost coincided

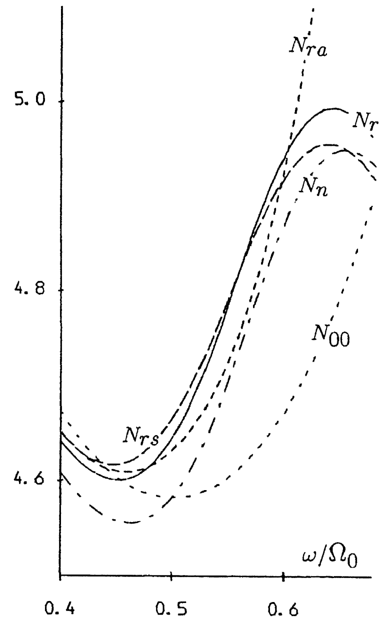

Figure 1. Plots of N_r versus $Y^{-1} \equiv \omega/\Omega_0$ (——) (see (8)), N_n versus $Y^{-1} \equiv \omega/\Omega_0$ (— · —) (see (10)), N_{ra} versus $Y^{-1} \equiv \omega/\Omega_0$ (- - - -) (see (13)), N_{00} versus $Y^{-1} \equiv \omega/\Omega_0$ (- - -) (see (12)) and N_{rs} versus $Y^{-1} \equiv \omega/\Omega_0$ (— — —) (see (17)) for a plasma with the following parameters: $r = 0.01$, $\nu = 5$ and $A_e = 2$. Plots are shown only for those $Y^{-1} \equiv \omega/\Omega_0$ for which $|\bar{\gamma}_{r,rs,n,ns}| \lesssim 0.2$ (i.e. for which condition (7) is satisfied).

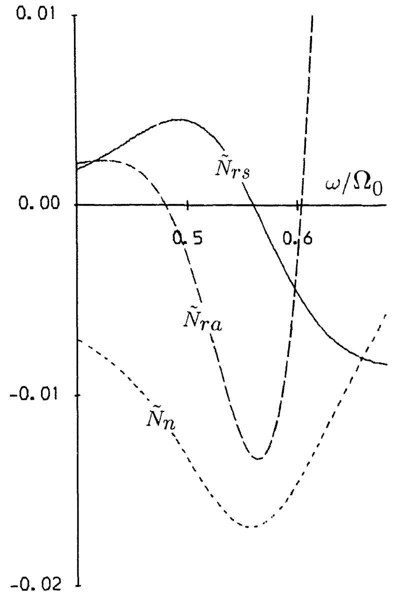

Figure 2. Plots of $\tilde{N}_{rs}$ versus $Y^{-1} \equiv \omega/\Omega_0$ (——) (see (19)), $\tilde{N}_n$ versus $Y^{-1} \equiv \omega/\Omega_0$ (- - - -) (see (20)), $\tilde{N}_{ra}$ versus $Y^{-1} \equiv \omega/\Omega_0$ (— — —) (see (21)) for a plasma with the same parameters as in figure 1.

with the curves $\bar{\gamma}_r$ (cf figures 2 to 6 in Sazhin *et al* 1992b). In what follows we will restrict ourselves to presenting the curves $\tilde{N}_{rs}$, $\tilde{N}_{rn}$ and $\tilde{N}_{ra}$ versus ω/Ω_0 for these five cases.

Plots of $\tilde{N}_{rs}$, $\tilde{N}_{rn}$ and $\tilde{N}_{ra}$ versus ω/Ω_0 for $r = 0.01$, $\nu = 1$ and $A_e = 2$ are shown in figure 4. As follows from this figure, the curve $\tilde{N}_{rs}$ can approximate $\tilde{N}_r$ with even greater accuracy than in the case shown in figure 2. The accuracy of approximation of N_r by N_{rs} is better than 0.5% at $\omega \lesssim 0.75\Omega_0$. At $\omega \lesssim 0.45\Omega_0$ there seems to be no advantage in using the approximation N_{rs} when compared with the much simpler approximation N_{ra}. At these frequencies both these approximations give an almost exact value of N_r.

Plots of $\tilde{N}_{rs}$, $\tilde{N}_{rn}$ and $\tilde{N}_{ra}$ versus ω/Ω_0 for $r = 0.01$, $\nu = 0.5$ and $A_e = 2$ shown in figure 5 are close to those shown in figure 4. The same set of curves for $r = 0.004$, $\nu = 1$ and $A_e = 2$ shown in figure 6 indicates further advantages in using the approximation N_{rs}. At $\omega \lesssim 0.84\Omega_0$ the possible error of approximation of N_r appears to be as low as 0.25%. At $\omega \lesssim 0.5\Omega_0$ the coincidence between the plots of N_r, N_{rs} and N_{ra} versus ω/Ω_0 is almost exact to within the accuracy of plotting.

In figure 7 we show the same set of curves as in figure 6 but for $A_e = 1$. Comparing these two figures one can see that the accuracy of approximation of N_r by N_{rs} increases with decreasing A_e, while the opposite takes place for the approximation

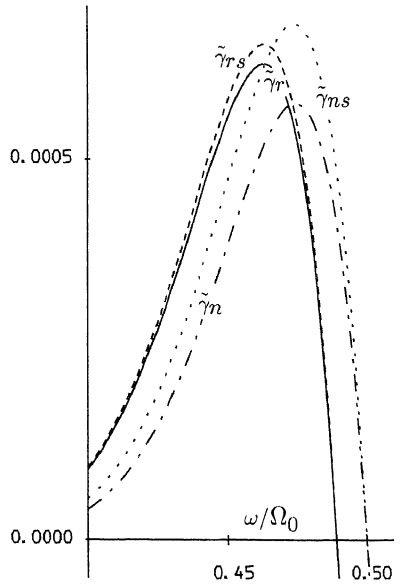

Figure 3. Plots of $\tilde{\gamma}_r$ versus $Y^{-1} \equiv \omega/\Omega_0$ (——) (see (9)), $\tilde{\gamma}_n$ versus $Y^{-1} \equiv \omega/\Omega_0$ (— · —) (see (11)), $\tilde{\gamma}_{ns}$ versus $Y^{-1} \equiv \omega/\Omega_0$ (- - -) (see (14)) and $\tilde{\gamma}_{rs}$ versus $Y^{-1} \equiv \omega/\Omega_0$ (— — —) (see (9) with $N = N_{rs}$) for the same plasma parameters as in figure 1.

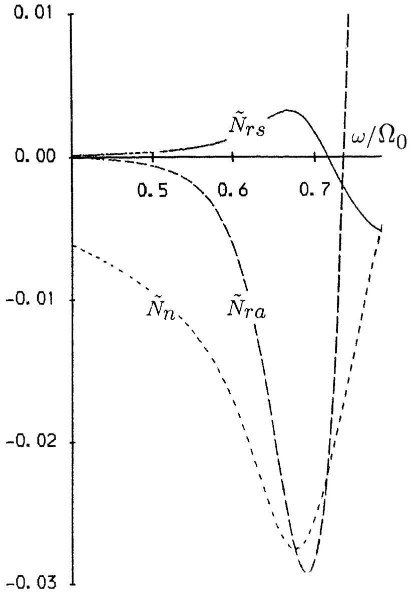

Figure 4. The same as figure 2 but for $\nu = 1$.

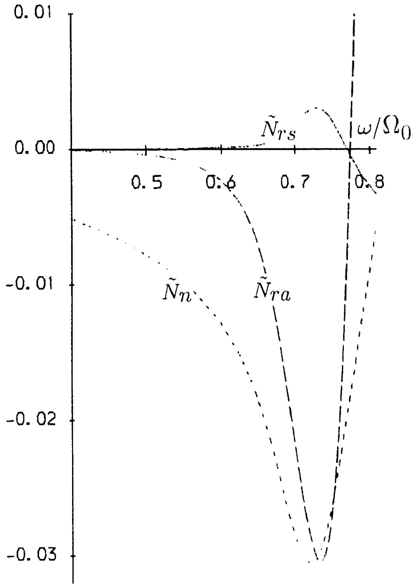

Figure 5. The same as figure 2 but for $\nu = 0.5$.

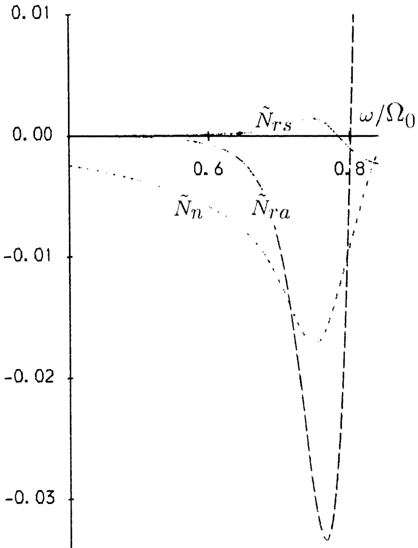

Figure 6. The same as figure 4 but for $r = 0.004$.

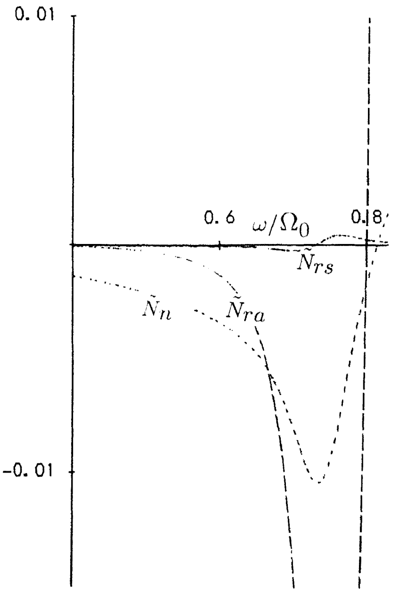

Figure 7. The same as figure 6 but for $A_e = 1$.

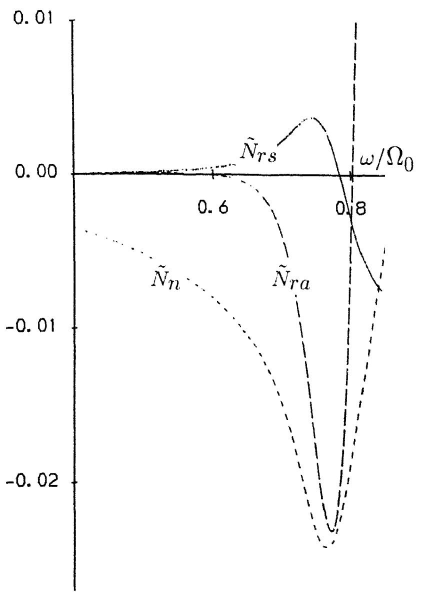

Figure 8. The same as figure 6 but for $A_e = 3$.

N_{ra}. At $\omega \lesssim 0.83\Omega_0$ the accuracy of approximation of N_r by N_{rs} appears to be as high as 0.05%.

In figure 8 we show the same set of curves as in figures 6 and 7 but for $A_e = 3$. In this case the accuracy of the approximation N_{rs} appears to be slightly lower when compared with the cases shown in figures 6 and 7, the error being lower than 0.8% at $\omega \lesssim 0.85\Omega_0$. However, even in this case the approximation N_{rs} appears to be noticeably better than the approximations N_{ra} and N_{rn}.

5. Conclusions

It is pointed out that the numerical solution of the real part of the parallel whistler-mode dispersion equation in a weakly relativistic plasma can be approximated by a perturbed solution of the real part of the corresponding non-relativistic dispersion equation. The obtained approximate solution for the real part of the wave refractive index, N, appeared to be resonably close (with possible error of a fraction of a percent) to the corresponding numerical solution of the weakly relativistic dispersion equation. The accuracy of this approximate solution is shown to increase with decreasing electron density, and/or electron energy, and/or the anisotropy of the electron distribution function. The advantages of our new solution are particularly apparent at relatively high frequencies when the cold plasma solution deviates considerably from the corresponding non-relativistic solution. However, at frequencies well below the electron gyrofrequency the new approximation seems to have no obvious advantages when compared with the solution based on the perturbation of the cold plasma solution.

Based on the obtained approximate value of N we calculated approximate values of the increment of wave growth or the decrement of their damping, γ, which appeared to be closer to the values of γ, as obtained by the exact numerical solution, than any previously obtained approximations (see Sazhin *et al* 1992b). Hence we can conclude that in many practically important cases the very time consuming numerical solution of the parallel weakly relativistic dispersion equation can be replaced by the approximate solution considered in this paper with a possible error of not more than a fraction of a percent. Our approximate solution is a very general one and can be applied to the analysis of whistler-mode waves both in laboratory and space plasma.

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